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GENERALIZED PERIODIC PROBLEM OF ELASTICITY THEORY FOR A CRACKED HALF-PLANE*

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The state of stress of an isotropic half-plane weakened by a regular system of rectilinear cracks perpendicular to its edge is studied. The formulation of the problem is different from that used in /1/ in that the load on the crack does not generally possess any periodic properties. This circumstance makes utilization of the approaches proposed in /1/ for the solution impossible in practice.

As is well-known, the problem reduces to infinite systems of singular integral equations in unknown jumps which the derivatives of the displacements undergo during the passage through each crack. To simplify the system obtained, a scheme of analytic accounting of the symmetry of the elastic geometric characteristics of the medium is used /2/, according to which the generalized periodic problems are studied first. As is shown in the paper, a system of four singular integral equations in the desired jumps on the fundamental crack corresponds to them. These jumps are determined by the method of orthogonal polynomials, and by using elementary algebraic relationships on the other cracks.

1. Formulation of the problem. The cracked half-plane under consideration (Fig.1) possesses a symmetry group $C_{1h}{}^1$ in which translations (shifts) T_m on the vectors $m\mathbf{a}$ ($\mathbf{a}$ is the basis vector) and reflections Θ_m in the planes Π_m $(m = 0, \pm 1, \pm 2, \ldots)$ occur. A segment occupied by the main crack is understood to be Γ or $\Gamma^{(0,0)}$. In its turn, the crack obtained from a fundamental translation T_m or reflection Θ_m occupies the segment $\Gamma^{(0,m)} = T_m\Gamma$ or $\Gamma^{(1,m)} = \Theta_m\Gamma$ $(m = 0, \pm 1, \pm 2, \ldots)$.

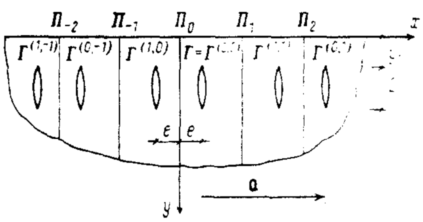

Fig.1

The geometry of the medium is determined by the length a of the main translation vector, the distance e from the main crack to the y axis (the Π_0 plane), and the ordinate y_1 of the upper end of the crack. Without loss of generality, the crack length is taken to be equal to two. The segment $\Gamma^{(j,m)}$ is removed a distance $e^{(j,m)}$ from the y axis, where $e^{(j,m)} = (-1)^j e + ma$ $(j = 0, 1)$.

The following conditions hold on the boundary of the medium

$$\sigma_y = 0, \ \sigma_{xy} = 0 \qquad (|x| < \infty, \ y = 0) \tag{1.1}$$

$$\sigma_x = -p^{(j,m)(1)}(y) \quad (x = e^{(j,m)}, \ y_1 \leqslant y \leqslant y_1 + 2, \ j = 0, \ 1) \tag{1.2}$$

$$\sigma_{xy} = -p^{(j,m)(2)}(y)$$

where σ_x, σ_y, and σ_{xy} are the normal and shear stresses, and $p^{(j,m)(1)}(y)$ and $p^{(j,m)(2)}(y)$ are given normal and tangential force functions acting on the crack boundary $\Gamma^{(j,m)}$ $(j = 0,1;\ m = 0, \pm 1, \pm 2, \ldots)$. We emphasize that according to (1.2) the forces applied to the different edges of one crack are identical, i.e., the jumps in the stresses σ_x and σ_{xy} are zero during passage through any crack. Such a constraint is not related to the possibilities of the proposed method of solution and is introduced exclusively from considerations of compactness of the subsequent calculations.

The biharmonic problem of elasticity theory with boundary conditions (1.1) and (1.2) reduces to an infinite system of singular integral equations (the asterisk denotes the absence of a component corresponding to the value $n = m = 0$ in the sum)

$$(-1)^{s+1}\int_{y_1}^{y_1+2}\chi^{(j,r)(s)}(\xi)\left[\frac{1}{\xi - y} + S(y,\xi)\right]d\xi + \tag{1.3}$$

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$$\sum_{k=1}^{2}\sum_{n=0}^{1}\sum_{m=-\infty}^{\infty}{}^{*}\int_{y_1}^{y_1+2}\chi^{(n,m)(k)}(\xi)R^{(j,r)(s)}_{(n,m)(k)}(y,\xi)\,d\xi=(-1)^s 4\pi p^{(j,r)(s)}(y)$$
$$(s=1,2;\ j=0,1;\ r=0,\pm1,\ldots;\ y_1\leqslant y\leqslant y_1+2).$$

Here

$$S(y,\xi)=\frac{1}{y+\xi}+\frac{2\xi}{(y+\xi)^2}-\frac{4\xi^2}{(y+\xi)^3}$$
$$R^{(j,r)(s)}_{(n,m)(k)}(y,\zeta)=R^{(s)}_{(k)}(e^{(n,m)}-e^{(j,r)},y,\zeta)$$
$$R^{(1)}_{(1)}(z,y,\zeta)=-(y-\zeta)[(y-\zeta)^2+3z^2]\Delta_-^{(2)}+2(y+\zeta)\Delta_+^{(1)}-(y+3\zeta)[(y+\zeta)^2-z^2]\Delta_+^{(2)}+4yz(y+\zeta)[(y+\zeta)^2-3z^2]\Delta_+^{(3)}$$
$$R^{(2)}_{(2)}(z,y,\zeta)=(y-\zeta)[(y-\zeta)^2-z^2]\Delta_-^{(2)}-(y-\zeta)[(y+\zeta)^2-z^2]\Delta_+^{(2)}-4y\zeta(y+\zeta)[(y+\zeta)^2-3z^2]\Delta_+^{(3)}$$
$$R^{(j)}_{(3-j)}(z,y,\xi)_-=z\{(-1)^j[(z^2-(y-\xi)^2)\Delta_-^{(2)}-\Delta_+^{(1)}+2(y-\xi+(-1)^j2\xi)(y+\xi)\Delta_+^{(2)}]+4y\xi[z^2-3(y+\xi)^2]\Delta_+^{(3)}\}.$$
$$(j=1,2;\ \Delta_\pm^{(k)}=[z^2+(y\pm\xi)^2]^{-k}).$$

We note that this system can also be constructed by the method of generalized integral transforms /3/.

The functions

$$\chi^{(j,m)(s)}(y)=\frac{1}{E}\left[\frac{\partial u_s}{\partial y}(e^{(j,m)}+0,y)-\frac{\partial u_s}{\partial y}(e^{(j,m)}-0,y)\right] \qquad (1.4)$$
$$(s=1,2;\ y_1\leqslant y\leqslant y_1+2)$$

are the unknowns in the system (1.3), i.e., the jumps which the derivatives of the displacement u_s undergo in passing through the segments $\Gamma^{(j,m)}$ $(j=0,1;\ m=0,\pm1,\pm2,\ldots)$ (u_1 and u_2 are understood to be the displacements along the x and y axes, respectively).

We note that the jumps mentioned should in addition satisfy the conditions that the crack be closed, which can be written in the form

$$\int_{y_1}^{y_1+2}\chi^{(j,r)(s)}(y)\,dy=0\quad(s=1,2;\ j=0,1;\ r=0,\pm1,\pm2,\ldots). \qquad (1.5)$$

Direct solution of system (1.3) is fraught with serious difficulties, hence it is best to simplify it because of the symmetry of the elastic geometric properties of the medium by using the well-known scheme in /2/. According to this, the problems under consideration can always be reduced, in practice, to generalized periodic problems.

2. Generalized periodic problems.

We shall consider the load to be described by any of two functions $p_{\alpha\rho}$ $(\rho=1,2)$ for which the forces $p_{\alpha\rho}^{(j,m)(s)}(y)$ act on the crack edges and satisfy the conditions

$$p_{\alpha\rho}^{(0,m)(s)}(y)=\sum_{\eta=1}^{2}\tau_{\alpha\rho\eta}(T_m)\,p_{\alpha\eta}^{(0,0)(s)}(y) \qquad (2.1)$$
$$p_{\alpha\rho}^{(1,m)(s)}(y)=(-1)^{s-1}\sum_{\eta=1}^{2}\tau_{\alpha\rho\eta}(\Theta_m)\,p_{\alpha\eta}^{(0,0)(s)}(y)$$
$$(s=1,2;\ m=0,\pm1,\pm2,\ldots;\ y_1\leqslant y\leqslant y_1+2)$$

where $\tau_{\alpha\rho\eta}(T_m)$ and $\tau_{\alpha\rho\eta}(\Theta_m)$ are elements on the intersection of the ρ-th row and η-th column of the matrix of two-dimensional representations (α is a scalar parameter, $|\alpha|\leqslant\pi$)

$$\tau_\alpha(T_m)=\begin{Vmatrix}\cos m\alpha & \sin m\alpha\\ -\sin m\alpha & \cos m\alpha\end{Vmatrix},\quad \tau_\alpha(\Theta_m)=\begin{Vmatrix}\cos m\alpha & -\sin m\alpha\\ -\sin m\alpha & -\cos m\alpha\end{Vmatrix}.$$

According to /2/, the components of the state of stress and strain that correspond to the loads mentioned, and therefore the unknown jumps $\chi_{\alpha\rho}^{(j,m)(s)}(y)$, possess properties similar to (2.1). In particular

$$\chi_{\alpha\rho}^{(0,m)(s)}(y)=\sum_{\eta=1}^{2}\tau_{\alpha\rho\eta}(T_m)\,\chi_{\alpha\eta}^{(0,0)(s)}(y) \qquad (2.2)$$

$$\chi_{\alpha\rho}^{(1,\,m)(s)}(y)=(-1)^{s-1}\sum_{\eta=1}^{2}\tau_{\alpha\rho\eta}(\Theta_m)\chi_{\alpha\eta}^{(0,\,0)(s)}(y)$$
$$(s=1,2;\ m=0,\pm1,\pm2,\ldots;\ \rho=1,2;\ y_1\leqslant y\leqslant y_1+2)\,.$$

Such problems are called generalized periodic problems. Their specific feature is that by virtue of (2.2) it is sufficient to find only the four jumps $\chi_{\alpha\rho}^{(0,\,0)(s)}(y)$ $(s=1,2;\rho=1,2)$. To do this just those integral equations from (1.3) can be used that correspond to the main crack, by writing them for each of the loads $p_{\alpha\rho}$. After taking account of Eqs.(2.2) the equations mentioned take the form

$$\begin{aligned}&(-1)^{s+1}\int_{y_1}^{y_1+2}\chi_{\alpha\rho}^{(0,\,0)(s)}(\xi)\left[\frac{1}{\xi-y}+S(y,\xi)\right]d\xi+\\&\sum_{k=1}^{2}\sum_{\eta=1}^{2}\sum_{m=-\infty}^{\infty}{}^{*}\tau_{\alpha\rho\eta}(T_m)\int_{y_1}^{y_1+2}\chi_{\alpha\eta}^{(0,\,0)(k)}(\xi)R_{(0,\,m)(k)}^{(0,\,0)(s)}(y,\xi)\,d\xi+\\&(-1)^{k+1}\sum_{m=-\infty}^{\infty}\tau_{\alpha\rho\eta}(\Theta_m)\int_{y_1}^{y_1+2}\chi_{\alpha\eta}^{(0,\,0)(k)}(\xi)R_{(1,\,m)(k)}^{(0,\,0)(s)}(y,\xi)\,d\xi=\\&(-1)^{s}4\pi p_{\alpha\rho}^{(0,\,0)(s)}(y)\quad(s=1,2;\ \rho=1,2;\ y_1\leqslant y\leqslant y_1+2)\end{aligned}\tag{2.3}$$

where the asterisk denotes the absence of a component corresponding to the value $m=0$ in the summation.

The remaining equations from (1.3) are satisfied automatically /2/.

The integral equations (2.3) of the generalized periodic problem are solved by the method of orthogonal polynomials /3/.

We will seek the desired functions $\chi_{\alpha\rho}^{(0,\,0)(s)}(\xi)$ in the form of the series

$$\chi_{\alpha\rho}^{(0,\,0)(s)}(\xi)=(1-t^2)^{-1/2}\sum_{r=0}^{\infty}X_{\rho r}^{(s)}T_r(t)\quad(s=1,2;\ \rho=1,2)\tag{2.4}$$

where $-1\leqslant t=\varphi(\xi)\leqslant1$, $\varphi(\xi)=\xi-y_1-1$, $T_r(t)$ are Chebyshev polynomials of the first kind, and $X_{\rho r}{}^{(s)}$ are scalar coefficients.

It immediately follows from conditions (1.5) that $X_{\rho 0}{}^{(s)}=0$. Consequently, the summation in (2.4) can start from the value $r=1$.

Substituting the expansion (2.4) into (2.3), we multiply both sides of the latter by $(1-g^2)^{1/2}U_{l-1}(g)$, where $l=1,2,\ldots$, $g=\varphi(y)$, $U_{l-1}(g)$ are Chebyshev polynomials of the second kind, and we integrate with respect to y between y_1 and y_1+2.

We make a change of variables by setting $y=\varphi^{-1}(g)$ and $\xi=\varphi^{-1}(t)$, where $\varphi^{-1}(z)=z+y_1-1$, and we take into account that

$$\langle(y-\xi)^{-1}\rangle=-2^{-1}\pi^2\delta_{lr}$$
$$\left(\langle f\rangle=\int_{-1}^{1}\sqrt{1-g^2}\,U_{l-1}(g)\,dg\int_{-1}^{1}\frac{T_r(t)}{\sqrt{1-t^2}}f\,[\varphi^{-1}(g),\varphi^{-1}(t)]\,dt\right)$$

where δ_{lr} is the Kronecker delta. We consequently arrive at an infinite algebraic system of equations in the desired coefficients

$$\begin{aligned}&\sum_{\eta=1}^{2}\sum_{k=1}^{2}\sum_{r=1}^{\infty}X_{\eta r}^{(k)}A_{\rho\eta lr}^{(s)(k)}=Y_{\rho l}^{(s)}\quad(\rho=1,2;\ s=1,2;\ l=1,2)\\&A_{\rho\eta lr}^{(s)(k)}=(-1)^{s-1}\delta_{\rho\eta}\delta_{ks}\delta_{lr}\left(\frac{\pi^2}{2}-\langle S\rangle\right)+\langle B_{sk}\rangle\\&B_{sk}(y,\xi)=\sum_{m=-\infty}^{\infty}{}^{*}\tau_{\alpha\rho\eta}(T_m)\,R_{(0,\,m)(k)}^{(0,\,0)(s)}(y,\xi)-\\&\qquad(-1)^{k+1}\sum_{m=-\infty}^{\infty}\tau_{\alpha\rho\eta}(\Theta_m)\,R_{(1,\,m)(k)}^{(0,\,0)(s)}(y,\xi)\\&Y_{\rho l}^{(s)}=(-1)^{s}4\pi\int_{-1}^{1}\sqrt{1-g^2}\,U_{l-1}(g)\,p_{\alpha\rho}^{(0,\,0)(s)}[\varphi^{-1}(g)]\,dg\end{aligned}\tag{2.5}$$

It is best to use known quadrature formulas in evaluating the integrals /4/.

The approximate solution of system (2.5) is constructed by using the method of reduction. Numerical analysis shows that conservation of the first nine components in expansion (2.4) ensures sufficient accuracy for practical applications over a broad range of geometric characteristics.

The solution of system (2.5), i.e., the evaluation of the unknowns $X_{\rho r}^{(s)}$ enables the desired jumps $\chi_{\alpha\rho}^{(j,m)(s)}$ $(s = 1, 2)$ to be found on each segment $\Gamma^{(j,m)}$ $(j = 0, 1;\ m = 0, \pm 1, \pm 2, \ldots)$ for a load $p_{\alpha\rho}$ $(\rho = 1, 2)$ by means of (2.4) and (2.2). All the required characteristics of the state of stress and strain of a half-plane are expressed in a known manner in terms of these jumps.

Let $K_{\alpha\rho(i)}^{(j,m)(s)}$ $(i = 1, 2)$ denote the normal $(s = 1)$ and shear $(s = 2)$ stress intensity factors at points with ordinates y_1 $(i = 1)$ and $y_1 + 2$ $(i = 2)$ of the segment $\Gamma^{(j,m)}$ for a load $p_{\alpha\rho}$. We can write /1/

$$K_{\alpha\rho(1)}^{(0,0)(s)} = \frac{1}{4}\sum_{r=1}^{\infty}(-1)^r X_{\rho r}^{(s)}, \quad K_{\alpha\rho(2)}^{(0,0)(s)} = -\frac{1}{4}\sum_{r=1}^{\infty} X_{\rho r}^{(s)} \tag{2.6}$$
$$(s = 1, 2;\ \rho = 1, 2).$$

The remaining intensity factors can be found from relationships analogous to (2.2).

The approach proposed for the numerical investigation of generalized periodic problems for a cracked half-plane is relatively simple and effective, as is seen from the above.

3. Scheme for the solution of non-periodic problems.

We assume that the load $p = p_\mu^{(2N)}$ given on a set of segments $\Gamma^{(j,m)}$ $(j = 0, 1;\ m = 0, \pm 1, \pm 2, \ldots)$ is arbitrary on segments between the planes Π_0 and Π_{2N} (N is a certain positive integer) and is symmetric $(\mu = 1)$ or skew-symmetric $(\mu = 2)$ with respect to all the planes Π_{2Nr} $(r = 0, \pm 1, \pm 2, \ldots)$. We let $p_\mu^{(j,m)(s)}$ denote their corresponding normal $(s = 1)$ and shear $(s = 2)$ forces on the edges of the crack occupying the segment $\Gamma^{(j,m)}$.

The following expansion /5/ then holds:

$$p_\mu^{(2N)} = \sum_{i=0}^{N}\frac{M_i}{2N} p_{\alpha_i,\mu}; \quad \alpha_i = \frac{i\pi}{N}, \quad M_0 = M_N = 1, \tag{3.1}$$
$$M_i = 2 \quad (0 < i < N)$$

where $p_{\alpha_i,\mu}$ is the μ-th of the functions $p_{\alpha_i,\rho}$ $(\rho = 1, 2)$ that generate functions $p_{\alpha_i,\rho}^{(j,m)(s)}$ $(s = 1, 2)$ interrelated by (2.1) and defined on the fundamental segment $\Gamma^{(0,0)}$ by the expressions

$$p_{\alpha_i,\rho}^{(0,0)(s)}(y) = \sum_{n=0}^{N-1}\tau_{\alpha_i\mu\rho}(T_n)\, p_\mu^{(0,n)(s)}(y) + (-1)^{s-1}\sum_{n=1}^{N}\tau_{\alpha_i\mu\rho}(\Theta_n)\, p_\mu^{(1,n)(s)}(y) \tag{3.2}$$
$$(s = 1, 2;\ \rho = 1, 2;\ y_1 < y < y_1 + 2)$$

in the segments $\Gamma^{(j,m)}$ $(j = 1, 2;\ m = 0, \pm 1, \pm 2, \ldots)$

By virtue of the above we propose a scheme for investigating the state of stress and strain of the cracked half-plane taking the load $p_\mu^{(2N)}$ into account. Initially the generalized periodic problems corresponding to the components $p_{\alpha_i,\mu}$ $(i = 0, 1, \ldots, N)$ of the expansion (3.1) should be solved. The functions $p_{\alpha_i,\rho}^{(0,0)(s)}$ needed to construct systems (2.5) are calculated from the inequalitites (3.2). Afterwards, the desired jumps denoted by $\chi_\mu^{(j,m)(s)}(y)$ can be determined by the superposition principle from the formulas

$$\chi_\mu^{(j,m)(s)}(y) = \sum_{i=0}^{N}\frac{M_i}{2N}\chi_{\alpha_i,\mu}^{(j,m)(s)}(y) \tag{3.3}$$
$$(s = 1, 2;\ j = 0, 1;\ m = 0, \pm 1, \pm 2, \ldots;\ \mu = 1, 2;\ y_1 < y \leqslant y_1 + 2)$$

Letting N tend to infinity, we arrive at the load $p_\mu = p_\mu^{(\infty)}$, which is arbitrary to the right of the Π_0 plane and symmetric $(\mu = 1)$ or skew-summetric $(\mu = 2)$ about this plane. If the condition

$$\lim_{N\to\infty}\Big[\sum_{n=0}^{N-1}|p_\mu^{(0,n)(s)}(y)| + \sum_{n=1}^{N}|p_\mu^{(1,n)(s)}(y)|\Big] < a \quad (y_1 \leqslant y \leqslant y_1 + 2) \tag{3.4}$$

is satisfied for the function p_μ, where a is a certain positive number, then by using the passage to the limit as $N \to \infty$, expressions (3.1) and (3.2) are transformed to a form enabling the proposed scheme to be extended to the load p_μ.

Actually, we have in place of (3.1) and (3.2)

$$p_\mu = \frac{1}{2\pi}\lim_{N\to\infty}\sum_{i=0}^{N} M_i p_{\alpha_i,\mu}\frac{\pi}{N} \quad (\mu = 1, 2) \tag{3.5}$$

$$p_{\alpha_i,\rho}^{(0,0)(s)}(y) = \lim_{N\to\infty}\Big[\sum_{n=0}^{N-1}\tau_{\alpha_i\mu\rho}(T_n)\, p_\mu^{(0,n)(s)}(y) - \tag{3.6}$$

$$(-1)^{s+1}\sum_{n=1}^{N}\tau_{\alpha_i\mu\rho}(\Theta_n)\,p_\mu^{(1,\,n)(s)}(y)]$$
$$(s=1,2;\ \rho=1,2;\ y_1\leqslant y\leqslant y_1+2).$$

Since $|\tau_{\alpha\mu\rho}(T_n)|\leqslant 1$ and $|\tau_{\alpha\mu\rho}(\Theta_n)|\leqslant 1$, it follows from (3.4),(3.6) and (2.1) that

$$|p_{\alpha_i,\rho}^{(j,\,m)(s)}(y)|<2a\quad(s=1,\ 2;\ \rho=1,\ 2;\ i=0,1,\ldots,N:\ y_1\leqslant y\leqslant y_1+2)$$

and therefore, Eq.(3.5) can be written in the form

$$p_\mu=\frac{1}{2\pi}\lim_{N\to\infty}\sum_{i=1}^{N-1}M_i p_{\alpha_i,\mu}\Delta\alpha<M_i a,\quad \Delta\alpha=\frac{\pi}{N-1}\,.$$

Remarking in this connection that $(i-1)\,\Delta\alpha<\alpha_i<i\Delta\alpha$ and $M_i\equiv 2\ (i=1,2,\ldots,N-1)$, we finally obtain

$$p_\mu=\frac{1}{\pi}\int_0^\pi p_{\alpha\mu}\,d\alpha\quad(\mu=1,2)\tag{3.7}$$

where in conformity with (3.6)

$$p_{\alpha\rho}^{(0,\,0)(s)}(y)=\sum_{n=0}^{\infty}\tau_{\alpha\mu\rho}(T_n)\,p_\mu^{(0,\,n)(s)}(y)+(-1)^{s+1}\sum_{n=1}^{\infty}\tau_{\alpha\mu\rho}(\Theta_n)\,p_\mu^{(1,\,n)(s)}(y)\tag{3.8}$$
$$(s=1,\ 2;\ \rho=1,2;\ y_1\leqslant y\leqslant y_1+2)\,.$$

By virtue of the superposition principle the desired jumps are

$$\chi_\mu^{(j,\,m)(s)}(y)=\frac{1}{\pi}\int_0^\pi \chi_{\alpha\mu}^{(j,\,m)(s)}(y)\,d\alpha\tag{3.9}$$
$$(s=1,2;\ \mu=1,2,\ j=0,1;\ m=0,\pm1,\pm2,\ldots).$$

For a numerical investigation of the state of stress and strain of a cracked half-plane corresponding to the load p_μ $(\mu=1,2)$, the integral relationship (3.9) is replaced by a finite combination of jumps $\chi_{\alpha_i,\mu}^{(j,\,m)(s)}$ $(i=0,1,\ldots,N)$ on the basis of a certain quadrature formula. The values of α_j are determined uniquely by the number N and the method of summation. Afterwards, the corresponding genralized periodic problems are solved for which the loads $p_{\alpha_i,\rho}^{(0,\,0)(s)}$ are calculated from (3.8).

We note that in the most general case the load should be decomposed into symmetric and skew-symmetric components. If each satisfies condition (3.4), the problem under consideration reduces to a generalized periodic problem even this time.

4. Internal pressure on the fundamental crack. We consider the state of stress and strain of a half-plane for which a uniformly distributed internal pressure of intensity q acts on the edges of the main crack, while the edges of the remaining cracks are force-free.

We decompose the load into symmetric and skew-symmetric parts with respect to the Π_0 plane. Obviously

Table 1

$a/2$	y_1	$(m,j)=(0,0)$ $i=1$	$(0,0)$ 2	(1.1) 1	(1.1) 2	(1.0) 1	(1.0) 2
1.5	0.5	138	128	−23	−27	−3	−2
	1	128	124	−26	−27	−3	−2
	3	122	121	−27	−27	−3	−3
	200	121	121	−27	−27	−3	−3
2	0.5	129	118	−15	−18	−5	−3
	1	118	115	−18	−19	−2	−2
	3	112	111	−19	−19	−3	−3
	200	110	110	−20	−20	−3	−3
4	0.5	123	111	−9	−6	−4	−3
	1	110	106	−6	−6	−3	−2
	3	103	103	−7	−7	−1	−1
	200	101	101	−7	−7	−2	−2
∞	0.5	120	110	—	—	—	—
	1	109	105	—	—	—	—
	3	104	102	—	—	—	—
	200	100	100	—	—	—	—

$$p_1^{(0,0)(1)}(y) = p_2^{(0,0)(1)}(y) = q/2,\ p_1^{(0,0)(2)}(y) = p_2^{(0,0)(2)}(y) = 0$$
$$p_\mu^{(j,m)(s)}(y) \equiv 0\ (s = 1, 2;\ j = 0, 1;\ \mu = 1, 2;\ m = 1, 2, \ldots).$$

It can be verified that condition (3.4) is satisfied. Therefore, the components mentioned are representable in the form (3.7), and their corresponding jumps $\chi_\mu^{(j,m)(s)}(y)$ can be evaluated by means of (3.9).

According to (3.8) we should set

$$p_{\alpha\rho}^{(0,0)(s)}(y) = \delta_{s1}\delta_{\rho\mu}q/2\ (s = 1, 2;\ \rho = 1, 2)$$

in the resolving Eqs.(2.5) of the generalized periodic problems, where $\mu = 1$ for the symmetric, and $\mu = 2$ for the skew-symmetric case.

The Simpson quadrature formula with eleven nodes was used to evaluate the integrals (3.9).

The desired state of stress and strain is determined by the superposition of the symmetric and skew-symmetric states. In particular, the coefficients $K_{(i)}^{(j,m)(s)}$ of the normal $(s = 1)$ and shear $(s = 2)$ stress intensities at the points with ordinates y_1 $(i = 1)$ and $y_1 + 2$ $(i = 2)$ of the segment $\Gamma^{(j,m)}$ are found as

$$K_{(i)}^{(j,m)(s)} = \sum_{\mu=1}^{2} K_{\mu(i)}^{(j,m)(s)} \quad (j = 0, 1;\ m = 0, \pm 1, \pm 2, \ldots)$$

where $K_{\mu(i)}^{(j,m)(s)}$ $(\mu = 1, 2)$ are the corresponding intensity factors for a load p_μ.

The quantities $10^2 \cdot K_{(i)}^{(j,m)(1)}/q$ are shown in Table 1 for three cracks for different values of the parameters $a/2$ and y_1. It is considered that $e = a/4$, i.e., the spacing between all the adjacent cracks is identical and equal to $a/2$.

For $a/2 = \infty$ the half-plane under consideration is weakened only by one (main) crack and the values of $K_{(i)}^{(0,0)(1)}$ $(i = 1, 2)$ agree with the data from /6/. As $y_1 \to \infty$ the values of $K_{(i)}^{(j,m)(1)}$ approach the values of the corresponding intensity factors in the problem of an isotropic plane weakened by a periodic system of cracks that is analogous in loading and geometry /1/.

The approach of the cracks to the half-plane boundary results, as usual, in magnification of the intensity factors. As regards the effect of closure of the cracks, the numerical analysis performed then enables a new qualitative deduction to be made.

As is known /1/. The closure of cracks loaded by internal pressure results in a decrease in the intensity factors. At the same time it turns out that the passage of unloaded cracks to a loaded crack causes the growth of these coefficients thereon.

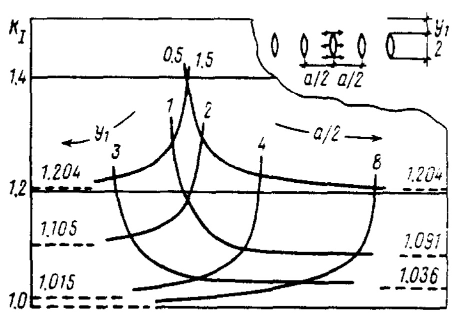

Fig.2

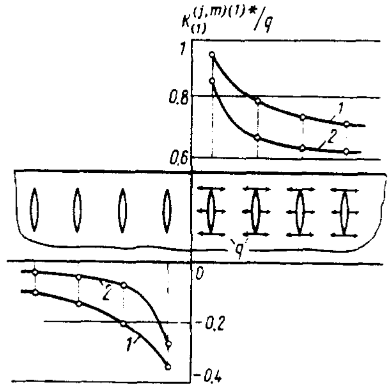

Fig.3

To make the above more graphic, a nomogram for the quantities $K_{(1)}^{(0,0)(1)}/q = K_1$ is presented in Fig. 2. It contains lines of two kinds. For each line of the first kind the parameter y_1 is fixed ($y_1 = 0.5, 1, 3$), while the parameter $a/2$ varies continuously. For lines of the second kind, on the other hand, the value of the parameter $a/2$ is fixed $a/2$ ($a/2 = 1.5, 2, 4, 8$). For given values of the parameters y_1 and $a/2$ the quantity $K_1^{(0,0)(1)}/q$ is determined as the ordinate of the intersection of corresponding lines of the first and second kinds. The asymptotes of the nomogram lines are shown by dashes in Fig. 2. Their physical meaning is obvious.

5. Other versions of crack loading by internal pressure. For any loading of cracks by uniform internal pressure the characteristics of the state of stress and strain of a

Table 2

a/2	$y_1 = 0.5$	1	3	200
2	345	294	263	263
4	200	173	114	114
8	69	65	37	34

half-plane can be obtained by a linear combination of the corresponding characteristics by solving the problem elucidated in Sec. 4.

As an illustration we select the case when all the cracks on the right of the Π_0 plane are loaded by internal pressure with identical intensity q and on the left are not loaded. We emphasize that this problem cannot be solved directly by using the method proposed, since condition (3.4) is not satisfied.

Keeping the same meaning for the notation $K_{(i)}^{(j,\, m)(s)}$ that it has in Sec. 4 and understanding $K_{(i)}^{(j,\, m)(s)*}$ to be the corresponding intensity factors in the new problem, we obtain on the basis of superposition

$$K_{(i)}^{(0,0)\,(1)*} = K_{(i)}^{(0,0)\,(1)} + \Delta_{(i)}^{(1)} \tag{5.1}$$

$$\Delta_{(i)}^{(1)} = \sum_{j=0}^{1} \sum_{m=1}^{\infty} K_{(i)}^{(j,\, m)\,(1)} \quad (i = 1, 2) .$$

Direct calculation of the quantities $\Delta_{(i)}^{(1)}$ is quite difficult for a number of reasons. Consequently, we will use a special method. The representation

$$K_{01\,(i)}^{(0,0)\,(1)} = K_{(i)}^{(0,0)\,(1)} + \sum_{j=0}^{1} \left[\sum_{m=1}^{\infty} K_{(i)}^{(j,\, m)\,(1)} + \sum_{m=1}^{\infty} K_{(i)}^{(j,\, -m+j)\,(1)} \right] \tag{5.2}$$

is valid in the ordinary periodic problem corresponding to a load p_{01}.

We shall consider $e = a/4$. Then by virtue of symmetry

$$K_{(i)}^{(j,\, -m+j)\,(1)} = K_{(i)}^{(j,\, m)\,(1)} \quad (j = 0,1;\ m = 1, 2, \ldots) .$$

Using the notation introduced in (5.1) for $\Delta_{(i)}^{(1)}$, we obtain from (5.2)

$$\Delta_{(i)}^{(1)} = (K_{01\,(i)}^{(0,0)\,(1)} - K_{(i)}^{(0,0)\,(1)})/2 . \tag{5.3}$$

The coefficients $K_{(i)}^{(0,\,0)(1)}$ were determined in Sec. 4, while the coefficients $K_{01(i)}^{(0,\,0)(1)}$ are found by solving the periodic problem. Consequently, Eq.(5.3) permits a relatively simple calculation of $\Delta_{(i)}^{(1)}$ for different values of the parameters $a/2$ and y_1. Certain data on the quantity $-10^3 \cdot \Delta_{(1)}^{(1)}$ are collected in Table 2.

The intensity factors on the fundamental crack are determined from (5.1), and on the other cracks by using obvious relationships. In particular

$$K_{(i)}^{(1,1)\,(1)*} = K_{(i)}^{(0,0)\,(1)*} + K_{(i)}^{(1,1)\,(1)}, \quad K_{(i)}^{(0,1)\,(1)*} = K_{(i)}^{(1,1)\,(1)*} + K_{(i)}^{(0,1)\,(1)}$$

$$K_{(i)}^{(1,0)\,(1)*} = \Delta_{(i)}^{(1)}, \quad K_{(i)}^{(0,\,-1)\,(1)*} = K_{(i)}^{(1,0)\,(1)*} - K_{(i)}^{(1,1)\,(1)}$$

$$K_{(i)}^{(1,\,-1)\,(1)*} = K_{(i)}^{(0,\,-1)\,(1)*} - K_{(i)}^{(0,1)\,(1)}, \ldots .$$

Conditional graphs of the change in the quantities $K_{(i)}^{(j,\, m)(1)*}/q$ $(i = 1, 2)$ with distance from the Π_0 plane are shown in Fig. 3. Lines 1 and 2 correspond, respectively, to the values $y_1 = 0.5$ and $y_1 = 3$. The parameter $a/2$ is taken equal to 2. The asymptotes of the graphs as $m \to -\infty$ agree with the abscissa axis and as $m \to \infty$ are associated with ordinary periodic problems.

Loads on cracks can be encountered in the most diverse combinations, hence, estimates of the magnitude of the maximum stress intensity factor are of considerable interest from the practical viewpoint. If uniformly distributed normal forces not exceeding q in absolute value act on the crack edges, then by elementary analysis it can be seen that an exact extimate holds: the stress intensity factors on any crack cannot exceed the value

$$K_{(1)}^{(0,0)\,(1)} + 2\Delta_{(1)}^{(1)} .$$

If the mentioned normal forces have the same sign on all the cracks (internal pressure), then the value $K_{(1)}^{(0,\,0)(1)}$ is a maximum (Fig. 2).

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BRITTLE CLEAVAGE OF A PIECEWISE-HOMOGENEOUS ELASTIC MEDIUM*

I.V. SIMONOV

Stationary pre-Rayleigh motion of a rigid body along a straight line connecting two elastic half-planes with the formation of a crack and a cavern is investigated. The contact between the edges in a small zone of the edge of the crack and outside the cavern at a large distance from the wedge is taken into account by the method of joining asymptotic expansions. As is shown, the ratios between the characteristic lengths are, respectively, quite small and quite large parameters if the wedge velocity is not close to the Rayleigh velocity, which specifies the advisability of using such an approach.

1. An absolutely blunt rigid wedge of thickness $h(x)$, $|x| \leqslant 1$ moves without friction at a constant velocity c along the interface $y=0$, $|x| < \infty$ of two elastic media occupying the half-plane $y > 0$ (medium 1) and $y < 0$ (medium 2) (Fig.1). A crack of length $a-1$ is formed ahead of the wedge and a cavity for $-\infty < x < -1$. The crack edges and the cavity do not interact and are force-free (an a priori assumption). The sides of the wedge are completely adjacent to to medium. Total contact conditions are satisfied for $x > a$, $y = 0$.

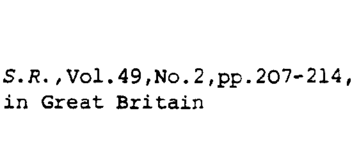

Fig.1

It is required to determine the steady stress field $\sigma_{km}{}^j(x, y)$ and the displacement field $U_m{}^j(x, y)$ from the following boundary conditions ($y = 0$):

$$U_{2,x}^j = h_j'(x) - \varphi,\ \sigma_{12}{}^j = 0,\ \sigma_{22}{}^j \leqslant 0,\ |x| < 1 \tag{1.1}$$
$$\sigma_{k2}{}^j = 0,\ [U_2] \geqslant 0,\ 1 < x < a \cup x < -1,\ [\sigma_{k2}] = [U_k] = 0,$$
$$x > a$$
$$[U_2(1)] = h(1),\quad \int_{-1}^{1} [\sigma_{22}] \left\| \begin{matrix} 1 \\ x \end{matrix} \right\| dx = 0 \quad (k, m, j = 1, 2)$$

Here $h_j = h_j(x)$ is the equation of the wedge surface relative to some of its axes, $h_j(x)$ are Hölder-continuous functions, $h = h_1 - h_2$, $h(1) \ll a - 1$, $|h_j'(x)| \ll 1$, $|x| < 1$, φ is the angle of rotation of the wedge axis, the subscript j defines the mediums, square brackets denote the jump in a quantity on passing from medium 1 into medium 2, the prime denotes ordinary differentiation, and the coordinate system is moving.

It is convenient to express the stresses and the derivatives of the displacements in dynamic linear elasticity theory (the plane problem, steady subsonic mode) in terms of analytic functions $\chi_m{}^j(z_{kj})$ of the complex variable $z_{kj} = x + i\beta_{kj}y$ by means of formulas /1/ (representations close to the representations in /2/). On the interface $z_{mj} = x$

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